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RADIALLY SYMMETRIC SOLUTIONS TO A SUPERLINEAR DIRICHLET PROBLEM IN A BALL WITH JUMPING NONLINEARITIES

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ABSTRACT. Let $p, \varphi: [0, T] \to R$ be bounded functions with $\varphi > 0$. Let $g: \mathbf{R} \to \mathbf{R}$ be a locally Lipschitzian function satisfying the superlinear jumping condition: (i) $\lim_{u\to-\infty}(g(u)/u) \in \mathbf{R}$, (ii) $\lim_{u\to\infty}(g(u)/u^{1+\rho}) = \infty$ for some $\rho > 0$, and (iii) $\lim_{u\to\infty}(u/g(u))^{N/2}(NG(\kappa u) - ((N-2)/2)u \cdot g(u)) = \infty$ for some $\kappa \in (0, 1]$ where G is the primitive of g. Here we prove that the number of solutions of the boundary value problem $\Delta u + g(u) = p(\|x\|) + c\varphi(\|x\|)$ for $x \in \mathbf{R}^N$ with $\|x\| < T$, $u(x) = 0$ for $\|x\| = T$, tends to $+\infty$ when c tends to $+\infty$. The proofs are based on the "energy" and "phase plane" analysis.

1. INTRODUCTION

In this paper we consider the existence of solutions to the Dirichlet problem

$$\begin{aligned} \Delta u + g(u) &= p(\|x\|) + c\varphi(\|x\|), \qquad x \in \Omega, \\ u &= 0, \qquad x \in \delta\Omega, \end{aligned} \tag{1.1}$$

where Ω is the ball of radius T in $\mathbf{R}^N$ centered at the origin, $g: \mathbf{R} \to \mathbf{R}$ is a locally Lipschitzian function, $p \in L^2(\Omega)$, $c \in \mathbf{R}$, and $\varphi: [0, T] \to \mathbf{R}$ is a differentiable function with

$$\varphi > 0 \quad \text{on } [0, T]. \tag{1.2}$$

In addition we assume that the problem is superlinear with jumping nonlinearities, i.e., that there exist real numbers M and $\rho > 0$ such that

$$\lim_{u\to-\infty} \frac{g(u)}{u} = M, \tag{1.3}$$

$$\lim_{u\to\infty} \frac{g(u)}{u^{1+\rho}} = \infty. \tag{1.4}$$

For the sake of simplicity of the proofs we assume, without loss of generality, that

$$\begin{aligned} &g(0) = 0 \text{ and } g \text{ is strictly increasing}, \ M > 0, \\ &p \in L^\infty(\Omega), \text{ and } 1 \le \varphi(t) \le 2 \text{ for all } t \in [0, T]. \end{aligned} \tag{1.5}$$

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In order to state our main results we introduce the following notations:

$$L(\kappa, u) = NG(\kappa u) - ((N-2)/2)ug(u), \tag{1.6}$$

$$L(\kappa) = \lim_{u\to\infty} L(\kappa, u)(u/g(u))^{N/2}, \tag{1.7}$$

where $G(u) = \int_0^u g(v)\,dv$ and $\kappa \in (0,1]$.

Our main result is

Theorem A. *If* (1.2)–(1.5) *hold and* $L(\kappa) = \infty$ *for some* $\kappa \in (0,1]$, *then there exists a positive integer* J *and an increasing sequence* $\{c_j : j = J, J+1, \ldots\}$ *tending to* ∞ *such that for* $c > c_j$ *the equation* (1.1) *has a radially symmetric solution* u_j *with* $u_j(0) < 0$, *and* u_j *has* j *interior nodal surfaces. In particular if* $c > c_j$ *then the equation* (1.1) *has* $j - J$ *radially symmetric solutions with* $u(0) < 0$.

Theorem A is in the spirit of studying boundary value problems for which the interval $(\lim_{u\to-\infty}(g(u)/u), \lim_{u\to\infty}(g(u)/u))$ contains at least one eigenvalue of $-\Delta$ with Dirichlet boundary conditions. Such a problem is called superlinear (resp. sublinear) if $\lim_{u\to\infty}(g(u)/u) = \infty$ (resp. $\lim_{u\to\infty}(g(u)/u) < \infty$). The one-dimensional superlinear version of Theorem A is given in [5], and it motivated our result (see also [4 and 13]). For studies on the sublinear case we refer the reader to [11] and references therein.

The proof of Theorem A is based on the shooting method that we have also used in [3]. We study the singular initial value problem

$$\begin{gathered} u'' + \frac{N-1}{t}u' + g(u) = p(t) + c\varphi(t), \qquad t \in [0, T], \\ u(0) = d, \qquad u'(0) = 0, \end{gathered} \tag{1.8}$$

where $d \in \mathbf{R}$. A simple argument based on the contraction mapping principle shows that (1.8) has a unique solution $u(t, d, c)$ on the interval $[0, T]$ depending continuously on (d, c) (see [3, Lemma 2.1]). Radially symmetric solutions of (1.1) are the solutions of (1.8) satisfying

$$u(T, d, c) = 0. \tag{1.9}$$

We analyze the energy of the corresponding solutions, i.e., we analyze the function

$$E(t, d, c) = \frac{(u'(t, d, c))^2}{2} + G(u(t, d, c)). \tag{1.10}$$

In order to count the number of zeros of the solutions to (1.8) we show that

$$E(t, -c^{\zeta}, c) > 0 \tag{1.11}$$

for $\zeta > (N+1)/(N+4)$ and c sufficiently large. In turn (1.11) implies that in the (u, u') plane, for c sufficiently large and for $d \le -c^{(N+1)/(N+4)}$, a continuous argument function $\psi(t, d, c)$ can be defined. The function $\psi(t, d, c)$ is such that $\psi(t,d,c) = i\pi + \pi/2$ (i an integer) if and only if $u(t,d,c) = 0$. We show that (see Lemma 4.1 and Lemma 2.7)

$$\limsup_{-d\to\infty} \psi(T,d,c) = S, \tag{1.12}$$

and that

$$\liminf_{c\to\infty} \psi(T, -c^{\xi}, c) = \infty, \tag{1.13}$$

where $S \in \mathbf{R}$ and $\xi \in [(N+1)/(N+4), 1)$. The intermediate value theorem and (1.13) imply that given any integer j there exists c_j such that if $c > c_j$ then

$$\psi(T, -c^{\xi}, c) \geq j\pi + \pi/2. \tag{1.14}$$

By combining (1.12), (1.14) and the intermediate value theorem we see that if c is sufficiently large and $j \geq J := [(S - \pi/2)/\pi] + 1$, then there exist numbers $d_J < d_{J+1} < \cdots < d_j$ with

$$\psi(t, d_i, c) = i\pi + \pi/2, \qquad i \in \{J, J+1, \ldots, j\}. \tag{1.15}$$

Hence

$$u_i(T, d_i, c) = 0, \qquad i \in \{J, J+1, \ldots, j\}. \tag{1.16}$$

Thus if c is sufficiently large, then (1.1) has $j - J$ radially symmetric solutions.

Remarks. (i) If Ω is a ring of the form $\{x \in \mathbf{R}^N : a \leq \|x\| \leq b\}$, then the equation in (1.8) is no longer singular and thus the problem reduces to the one studied in [5].

(ii) Condition (1.2) can be considerably weakened. For example, it is easy to verify that if $\varphi > 0$ on $[0, \varepsilon)$ and $\varphi = 0$ on $[\varepsilon, T]$, then Theorem A holds.

2. Phase-plane analysis

Let $(u(t,d,c), u'(t,d,c)) \neq (0,0)$ for all $t \in [0, \underline{t})$. By defining $r^2(t,d,c) = u^2(t,d,c) + (u'(t,d,c))^2$ we see that for $d < 0$ there exists a unique continuous argument function $\psi(t,d,c)$, $t \in [0,\underline{t})$, such that

$$\begin{aligned} u(t,d,c) &= -r(t,d,c)\cos\psi(t,d,c), \\ u'(t,d,c) &= r(t,d,c)\sin\psi(t,d,c), \\ \psi(0,d,c) &= 0. \end{aligned} \tag{2.1}$$

An elementary calculation shows that

$$\psi'(t,d,c) = \sin^2\psi(t,d,c) - \frac{(g(u(t,d,c)) + \frac{N-1}{t}u'(t,d,c) - c\varphi(t) - p(t))\cos\psi(t,d,c)}{r(t,d,c)}.$$

From this formula follows:

Remark 2.1. If $r(t,d,c) > 0$ for all $t \in [0,T]$, and $\psi(\hat{t},d,c) = (2k+1)\pi/2$ for some $\hat{t} \in [0,T]$ and some integer k, then $\psi(t,d,c) > (2k+1)\pi/2$ for all $t > \hat{t}$. In particular, $\psi(t,d,c) > -\pi/2$ for all $t \in [0,T]$.

Now let γ be such that

$$\gamma_1 := 1 - 12\varepsilon \le \gamma \le 1 - 10\varepsilon := \gamma_2, \tag{2.2}$$

where ε is defined by

$$\varepsilon = \frac{\rho}{4(6N + 6N\rho + 8 + 8\rho)}. \tag{2.3}$$

Elementary calculations show that

$$\begin{gathered}
\text{(a)}\quad 2\gamma > 2\gamma + (\gamma-1)N > 2\gamma + \frac{3}{2}(\gamma-1)N > 1 + \frac{1}{1+\rho}, \\
\text{(b)}\quad \gamma + \varepsilon < 1, \qquad \text{(c)}\quad \gamma > \varepsilon + \frac{1}{1+\rho}, \\
\text{(d)}\quad 3\gamma - 2 - \frac{\varepsilon}{2} > \frac{2\gamma}{2+\rho}, \qquad \text{(e)}\quad 2\gamma - 1 > \frac{N+1}{N+4}.
\end{gathered} \tag{2.4}$$

Let $u(t,d,c) := u(t)$ be a solution to (1.8). Since the following constant appears often in this section we let

$$Q := Q(N) := \frac{[\sqrt{2N} + 512N]^2}{2N}. \tag{2.5}$$

Throughout the paper k will denote a nonnegative integer.

Lemma 2.2. *There exists C_1 such that if $c > C_1$, $E(t_1) = c^{2\gamma}$, $u(t_1) > -\alpha c^{2\gamma-1}$ and $\psi(t_1) \in [2k\pi, 2k\pi + \pi/2]$, for some $t_1 \in (0,T]$ and $\alpha \in [1, 64N]$, then there exists $t_2 \in [t_1, t_1 + 2\alpha\sqrt{N}c^{\gamma-1}]$, such that*

$$\psi(t_2) = 2k\pi + \pi/2,$$

and

$$\frac{1}{128}c^{2\gamma} \le E(t_2) \le \frac{(\sqrt{2N} + 8\alpha)^2}{2N}c^{2\gamma} \le Qc^{2\gamma}.$$

In particular $E(t_2) = c^{2\gamma'}$ with $\gamma' \in [\gamma - (\ln 128)/(2\ln c), \gamma + (\ln Q)/(2\ln c)]$.

Proof. Let $t > t_1$ be such that $u(s) \le 0$ for all $s \in (t_1, t]$. Therefore, we have

$$\begin{aligned}
u'(t) &= t^{-N+1}\left[t_1^{N-1}u'(t_1) + \int_{t_1}^{t} s^{N-1}(c\varphi(s) + p(s) - g(u(s)))\,ds\right] \\
&\ge t^{-N+1}\int_{t_1}^{t} s^{N-1}(c - \|p\|_\infty)\,ds \\
&\ge \frac{c - \|p\|_\infty}{N}\left[t - \frac{(t_1)^N}{t^{N-1}}\right] \ge \frac{c - \|p\|_\infty}{N}(t - t_1).
\end{aligned}$$

Thus, for c sufficiently large we infer

$$\begin{aligned}
u(t) &\ge -\alpha c^{2\gamma-1} + \frac{c - \|p\|_\infty}{2N}(t - t_1)^2 \\
&\ge -\alpha c^{2\gamma-1} + \frac{c}{4N}(t - t_1)^2.
\end{aligned} \tag{2.6}$$

This shows that for some $t_2 \in [t_1, t_1 + 2\alpha\sqrt{N}c^{\gamma-1}]$ we have

$$u(t_2) = 0, \tag{2.7}$$

and $u'(s) \geq 0$ for all $s \in (t_1, t_2)$. In particular

$$\psi(t_2) = 2k\pi + \pi/2. \tag{2.8}$$

By the continuity of u' there exists $\tau < t_1$ such that $u'(\tau) = 0$. Since $u(t_1) \geq -\alpha c^{2\gamma-1}$, then from (1.3) for u sufficiently large, we have that $|g(u)| \leq (M+1)|u|$. Therefore, for c sufficiently large

$$G(u(t_1)) \leq [(M+2)/2]\alpha^2 c^{4\gamma-2}. \tag{2.9}$$

Thus for c sufficiently large we have

$$u'(t_1) \geq c^\gamma. \tag{2.10}$$

Hence,

$$c^\gamma \leq u'(t_1) = t_1^{-N+1}\int_\tau^{t_1} s^{N-1}(c\varphi(s) + p(s) - g(u(s)))\,ds \leq \frac{4}{N}c(t_1 - \tau),$$

which implies that

$$t_1 - \tau \geq Nc^{\gamma-1}/4. \tag{2.11}$$

Therefore,

$$\begin{aligned} u'(t_2) &= t_2^{-N+1}\int_\tau^{t_2} s^{N-1}(c\varphi(s) + p(s) - g(u(s)))\,ds \\ &\geq \frac{c - \|p\|_\infty}{N}(t_2 - \tau) \geq \frac{c}{8}c^{\gamma-1}. \end{aligned}$$

Thus

$$E(t_2) \geq \frac{1}{128}c^{2\gamma}. \tag{2.12}$$

Now, for $t \in (t_1, t_2]$ we have

$$\begin{aligned} u'(t) &= t^{-N+1}\left[t_1^{N-1}u'(t_1) + \int_{t_1}^t s^{N-1}(c\varphi(s) + p(s) - g(u(s)))\,ds\right] \\ &\leq \sqrt{2}c^\gamma + t^{-N+1}\int_{t_1}^t s^{N-1}(2c + \|p\|_\infty + (M+1)|u(t_1)|)\,ds \\ &\leq \sqrt{2}c^\gamma + t^{-N+1}\int_{t_1}^t s^{N-1}(2c + \|p\|_\infty + (M+1)\alpha c^{2\gamma-1})\,ds \\ &\leq \sqrt{2}c^\gamma + (2c + \|p\|_\infty + (M+1)\alpha c^{2\gamma-1})2\alpha\frac{\sqrt{N}}{N}c^{\gamma-1} \\ &\leq \sqrt{2}c^\gamma + 4c\frac{2\alpha}{\sqrt{N}}c^{\gamma-1} \leq \frac{\sqrt{2N} + 8\alpha}{\sqrt{N}}c^\gamma, \end{aligned} \tag{2.13}$$

where we have used (1.3), the hypothesis of the lemma and the fact that $t_2 \in [t_1, t_1 + 2\alpha\sqrt{N}c^{\gamma-1}]$. Since $E(t_2) = (u'(t_2))^2/2$ we see that for $c > C_1$ we have

$$E(t_2) \leq \frac{(\sqrt{2N} + 8\alpha)^2}{2N} c^{2\gamma}, \tag{2.14}$$

where C_1 is such that (2.6), (2.9) and (2.10) hold. This together with (2.8) and (2.12) proves the lemma.

Lemma 2.3. *If $c > C_2$, $\psi(\tau) = 2k\pi$, $u(\tau) = -c^{2\gamma-1}$, $u'(\tau) = 0$, with $\tau \in [0, T]$, then there exists $t_2 \in (\tau, \tau + 2\sqrt{N}c^{\gamma-1})$ such that*

$$\psi(t_2) = 2k\pi + \pi/2,$$

and

$$\frac{1}{16N} c^{2\gamma} \leq E(t_2) \leq \frac{(\sqrt{2N} + 8)^2}{2N} c^{2\gamma} \leq Q c^{2\gamma}.$$

In particular $E(t_2) = c^{2\gamma'}$ with $\gamma' \in [\gamma - (\ln 16N)/(2\ln c), \gamma + (\ln Q)/(2\ln c)]$.

Proof. Using the same arguments leading to the proof of (2.7) in Lemma 2.2 we get the existence of $t_2 \in [\tau, \tau + 2\sqrt{N}c^{\gamma-1}]$ satisfying $u(t_2) = 0$. Since for $u < 0$ sufficiently large $|g(u)| \leq (M+1)|u| \leq (M+1)c^{2\gamma-1}$, on $[\tau, t_2]$ we have

$$u'(t) \leq (3c + (M+1)c^{2\gamma-1}) \frac{1}{N} t \leq \frac{4}{N} c(t - \tau).$$

Thus,

$$u(t) \leq -c^{2\gamma-1} + \frac{2}{N} c(t-\tau)^2.$$

Hence, $t_2 - \tau \geq \sqrt{N}c^{\gamma-1}/\sqrt{2}$, which leads to $u'(t_2) \geq c^{\gamma}/2\sqrt{2N}$, and

$$E(t_2) \geq \frac{1}{16N} c^{2\gamma}. \tag{2.15}$$

On the other hand by replacing t_1 with τ in (2.11) and using the fact that $u'(\tau) = 0$, we obtain

$$E(t_2) \leq \frac{(\sqrt{2N} + 8)^2}{2N} c^{2\gamma}. \tag{2.16}$$

Hence, the lemma is proven.

Lemma 2.4. *There exists C_3 such that if $c > C_3$, $E(t_2) = c^{2\gamma}$, and $\psi(t_2) \in [2k\pi + \pi/2, 2k\pi + \pi)$ for some $t_2 \in [Nc^{\gamma-1}/4\sqrt{Q}, T]$, then there exists $t_3 \in (t_2, t_2 + 2c^{\gamma-1})$ such that $\psi(t_3) = 2k\pi + \pi$, and*

$$\tfrac{3}{4} c^{2\gamma} \leq E(t_3) \leq 2c^{2\gamma}.$$

In particular $E(t_3) = c^{2\gamma'}$ with $\gamma' \in [\gamma + (\ln \frac{3}{4})/(2\ln c), \gamma + (\ln 2)/(2\ln c)]$.

Proof. Without loss of generality we can assume that $u(t_2) \leq g^{-1}(4c)$. For $t > t_2$ such that $u(t) \leq g^{-1}(4c)$, and $\psi(s) \in [2k\pi + \pi/2, 2k\pi + \pi]$, for all

$s \in (t_2, t)$, we have

$$(2.17)\qquad \begin{aligned} E(t) &= E(t_2) + \int_{t_2}^{t} \left(-\frac{N-1}{s} u'(s) + c\varphi(s) + p(s) \right) u'(s)\, ds \\ &\le c^{2\gamma} + 3cu(t). \end{aligned}$$

Since

$$(2.18)\qquad G(u(t)) \le (4c) g^{-1}(4c) \le 16 c^{1+(1/(1+\rho))}$$

for c sufficiently large, from (2.17) using (2.4) we see that there exists a positive constant $\bar{\kappa}$ independent of (c, γ) such that $(u'(t))^2 \le \bar{\kappa}^2 c^{2\gamma}$. On the other hand

$$(2.19)\qquad \begin{aligned} E(t) &\ge E(t_2) + \int_{t_2}^{t} \left(-\frac{4(N-1)\sqrt{Q}\bar{\kappa}}{c^{\gamma-1}N} c^{\gamma} \right) u'(s)\, ds \\ &\ge c^{2\gamma} - \frac{4(N-1)\sqrt{Q}\bar{\kappa}}{N} cu(t) \\ &\ge c^{2\gamma} - \frac{4(N-1)\sqrt{Q}\bar{\kappa}}{N} c^{1+(1/(1+\rho))} \ge \frac{3}{4} c^{2\gamma}, \end{aligned}$$

for c sufficiently large (see (2.4)). From (2.18) and (2.19) we have

$$(u'(t))^2 \ge \tfrac{3}{4} c^{2\gamma}.$$

Thus

$$(2.20)\qquad (4c)^{1/(1+\rho)} \ge u(t) \ge \frac{\sqrt{3}}{2} c^{\gamma} (t - t_2).$$

Hence, there exists $t^* \in (t_2, t_2 + 6c^{(1/(1+\rho))-\gamma})$ such that

$$(2.21)\qquad u(t^*) = g^{-1}(4c) \quad \text{and} \quad u'(t^*) \ge \frac{\sqrt{3}}{2} c^{\gamma}.$$

For $t > t^*$ with $u'(s) \ge 0$ for all $s \in [t^*, t]$ we have

$$u''(t) = -\frac{N-1}{t} u'(t) + c\varphi(t) + p(t) - g(u(t)) \le -c.$$

Hence,

$$(2.22)\qquad u'(t) \le u'(t^*) - c(t - t^*) \le \frac{\sqrt{3}}{2} c^{\gamma} - c(t - t^*).$$

Thus, there exists $t_3 \in (t^*, t^* + \sqrt{3} c^{\gamma-1}/2) \subset (t_2, t_2 + 2c^{\gamma-1})$, (see (2.4)), such that

$$(2.23)\qquad u'(t_3) = 0$$

for c sufficiently large, and $u'(t) > 0$ for all $t \in [t^*, t_3)$. In particular

$$(2.24)\qquad \psi(t_3) = 2k\pi + \pi.$$

Moreover, for $t \in [t_2, t_3]$ from (2.17) we have

$$(2.25)\qquad G(u(t)) \le E(t) \le c^{2\gamma} + 3cu(t).$$

Thus from (1.4) we infer

$$(u(t))^{2+\rho} - 3cu(t) \leq c^{2\gamma}. \tag{2.26}$$

Now, if $(u(t))^{2+\rho} < 6cu(t)$, then

$$u(t) < 6c^{1/(1+\rho)}. \tag{2.27}$$

For $(u(t))^{2+\rho} \geq 6cu(t)$ from (2.26) we see that

$$u(t) \leq \tfrac{1}{3}c^{2\gamma-1}. \tag{2.28}$$

By using (2.4), (2.27), and (2.28) for c sufficiently large we infer

$$u(t) \leq \tfrac{1}{3}c^{2\gamma-1}. \tag{2.29}$$

Thus, replacing (2.29) in (2.25), for $t \in [t_2, t_3]$ we obtain

$$E(t) \leq 2c^{2\gamma}. \tag{2.30}$$

Hence, from (2.30) we have

$$u'(t) \leq c^{\gamma}. \tag{2.31}$$

Imitating the arguments leading to (2.19) it is easy to see that

$$E(t_3) \geq \tfrac{3}{4}c^{2\gamma}, \tag{2.32}$$

for $c > C_3$, where C_4 is assumed to be such that (2.19), (2.23), (2.29) and (2.32) hold. Thus, (2.24), (2.30) and (2.32) prove the lemma.

Lemma 2.5. *There exists C_4 such that if $c > C_4$, $E(t_3) = c^{2\gamma}$, and $\psi(t_3) \in [2k\pi + \pi, 2k\pi + \frac{3}{2}\pi]$ for some $t_3 \in [Nc^{\gamma-1}/8\sqrt{Q}, T]$ then there exists $t_4 \in [t_3, t_3 + 2c^{-\varepsilon/2}]$ such that*

$$\psi(t_4) = 2k\pi + \tfrac{3}{2}\pi.$$

Moreover,

$$\tfrac{3}{4}c^{2\gamma} \leq E(t_4) \leq c^{2\gamma}.$$

In particular $E(t_4) = c^{2\gamma'}$ with $\gamma' \in [\gamma + (\ln \frac{3}{4})/(2\ln c), \gamma]$.

Proof. Since $\psi(t_3) \in [2k\pi + \pi, 2k\pi + \frac{3}{2}\pi]$, we have $u(t_3) > 0$ and $u'(t_3) \leq 0$. Hence either

$$u(t_3) > g^{-1}(2c + \|p\|_\infty + c^{\gamma+\varepsilon}), \tag{2.33}$$

or

$$u(t_3) \leq g^{-1}(2c + \|p\|_\infty + c^{\gamma+\varepsilon}). \tag{2.34}$$

If (2.33) holds then for $t \geq t_3$ with $u(s) \geq g^{-1}(2c + \|p\|_\infty + c^{\gamma+\varepsilon})$ for all $s \in [t_3, t)$ we have

$$\begin{aligned} u'(t) &= t^{-N+1}\left[t_3^{N-1}u'(t_3) + \int_{t_3}^{t} s^{N-1}(c\varphi(s) + p(s) - g(u(s)))\,ds\right] \\ &\leq t^{-N+1}\int_{t_3}^{t} s^{N-1}(-c^{\gamma+\varepsilon})\,ds \\ &= \frac{-c^{\gamma+\varepsilon}}{N} - [t - (t_3^N)/t^{N-1}] \leq 0. \end{aligned} \tag{2.35}$$

Suppose that for all $t \in (t_3, t_3(1+c^{-\varepsilon/2}))$ we have $u(t) > g^{-1}(2c+\|p\|_\infty+c^{\gamma+\varepsilon})$. Hence, now for all $\lambda \in (-\varepsilon, -\varepsilon/2)$, by (2.35) we infer

$$\begin{aligned} u'(t_3(1+c^\lambda)) &\le \frac{-c^{\gamma+\varepsilon}}{N} t_3[(1+c^\lambda) - (1+c^\lambda)^{-N+1}] \\ &\le \frac{-c^{\gamma+\varepsilon}}{2^{N-1}} \frac{N}{8\sqrt{Q}} c^{\gamma-1} c^\lambda \le -\frac{N}{2^{N+2}\sqrt{Q}} c^{2\gamma-1}. \end{aligned} \tag{2.36}$$

Thus

$$\begin{aligned} u(t_3(1+c^{-\varepsilon/2})) &\le u(t_3(1+c^{-\varepsilon})) + \int_{t_3(1+c^{-\varepsilon})}^{t_3(1+c^{-\varepsilon/2})} -c^{2\gamma-1}[N/(2^{N+2}\sqrt{Q})]\,ds \\ &\le u(t_3) - \frac{N^2}{2^{N+5}Q} c^{3\gamma-2}(c^{-\varepsilon/2} - c^{-\varepsilon}). \end{aligned} \tag{2.37}$$

Since $E(t_3) = c^{2\gamma}$, we have that $u(t_3) \le G^{-1}(c^{2\gamma})$. By (1.4) we see that for c sufficiently large $g^{-1}(c^{2\gamma}) < c^{2\gamma/(1+\rho)}$. Hence, there exists $C_4 \ge \|p\|_\infty$ such that for $c > C_4$

$$u(t_3) \le c^{2\gamma/(2+\rho)}. \tag{2.38}$$

Also, since $\varepsilon > 0$ and (2.4) holds, we can assume C_4 to be such that $c^{-\varepsilon/2} - c^{-\varepsilon} > c^{-\varepsilon/2}/2$ for $c > C_4$, and that there exists a constant $k_1 > 0$ such that for $c > C_4$

$$u(t_3(1+c^{-\varepsilon/2})) \le c^{2\gamma/(2+\rho)} - k_1 c^{3\gamma-2-(\varepsilon/2)} \le 0, \tag{2.39}$$

which is a contradiction. Thus for $c > C_4$ there exists $t' \in (t_3, t_3(1+c^{-\varepsilon/2}))$ such that

$$u(t') = g^{-1}(2c + \|p\|_\infty + c^{\gamma+\varepsilon}). \tag{2.40}$$

This and (2.34) show that there exists $t'' \in [t_3, t_3(1+c^{-\varepsilon/2})]$ with

$$u(t'') \le g^{-1}(2c + \|p\|_\infty + c^{\gamma+\varepsilon}). \tag{2.41}$$

Now we estimate $E(t)$ for $t > t_3$ with $u'(s) \le 0$ for all $s \in [t_3, t]$. Since

$$\begin{aligned} \frac{dE(t)}{dt} &= -\frac{N-1}{t}(u'(t))^2 + (c\varphi(t) + p(t))u'(t) \\ &\le (c - \|p\|_\infty)u'(t) \le 0, \end{aligned} \tag{2.42}$$

we have

$$E(t) \le c^{2\gamma}. \tag{2.43}$$

Therefore,

$$|u'(t)| \le \sqrt{2}c^\gamma. \tag{2.44}$$

Hence,

$$-\frac{N-1}{t}u'(t) + c\varphi(t) + p(t) \le 4c, \tag{2.45}$$

for c sufficiently large. If in addition $u(t) \geq 0$ then

$$\begin{aligned} E(t) &= E(t_3) + \int_{t_3}^{t} \left(-\frac{N-1}{t}u'(s) + c\varphi(s) + p(s)\right) u'(s)\, ds \\ &\geq c^{2\gamma} + 4c(u(t) - u(t_3)) \geq c^{2\gamma} - 4cu(t_3) \\ &\geq c^{2\gamma} - 4c^{1+(2\gamma/(2+\rho))} \geq \tfrac{3}{4}c^{2\gamma}, \end{aligned} \tag{2.46}$$

where we have used (2.4) and (2.39). Thus from (2.43) and (2.44) we obtain

$$\tfrac{3}{4}c^{2\gamma} \leq E(t_2) \leq c^{2\gamma}. \tag{2.47}$$

Now, for $t > t''$ we have

$$\begin{aligned} G(u(t)) &\leq ug(u) \leq g^{-1}(4c)4c \\ &\leq 16c^{1+(1/(1+\rho))} \leq \tfrac{1}{4}c^{2\gamma}, \end{aligned} \tag{2.48}$$

(see (2.4)). Combining (2.47) and (2.48) we get

$$u'(t) < -c^{\gamma} \tag{2.49}$$

for all $t \geq t''$ such that $u(s) \geq 0$ and $u'(s) \leq 0$ with $s \in [t'', t]$. Hence

$$\begin{aligned} u(t) &\leq g^{-1}(2c + \|p\|_{\infty} + c^{\gamma+\varepsilon}) - c^{\gamma}(t - t'') \\ &\leq 4c^{1/(1+\rho)} - c^{\gamma}(t - t''), \end{aligned} \tag{2.50}$$

(see (2.4)) for c sufficiently large. From (2.50) we see that for some $t_4 \in [t'', t'' + c^{(1/(1+\rho))-\gamma}] \subset [t_3, t_3 + 2c^{-\varepsilon/2}]$ (see (2.4)) we have

$$u(t_4) = 0, \tag{2.51}$$

in particular $\psi(t_4) = 2k\pi + \frac{3}{2}\pi$ for $c > C_4$, where C_4 is in addition assumed to be such that (2.46)–(2.50) hold. This and (2.47) prove the lemma.

Lemma 2.6. *There exists C_5 such that if $c > C_5$, $\psi(t_4) \in [2k\pi + \frac{3}{2}\pi, 2(k+1)\pi]$, $E(t_4) = c^{2\gamma}$, with $t_4 \in [Nc^{\gamma-1}/8\sqrt{Q}, T]$, and $u(t_4) > -\alpha c^{2\gamma-1}$, $\alpha \in [1, 64N]$, then for some $t_5 \in [t_4, t_4 + \sqrt{2}c^{\gamma-1}]$ we have $\psi(t_5) = 2(k+1)\pi$. Moreover,*

$$-(\alpha + 4)c^{2\gamma-1} \leq u(t_5) \leq -\frac{1}{64\sqrt{Q} + 16}c^{2\gamma-1}.$$

Proof. Let now $t > t_4$ be such that $u'(s) \leq 0$ for all $s \in (t_4, t)$. The existence of such t's follows from the fact that $\psi(t_4) \in [2k\pi + \frac{3}{2}\pi, 2(k+1)\pi]$. Since

$$u''(s) = -\frac{N-1}{s}u'(s) + c\varphi(s) + p(s) - g(u(s)) \geq \tfrac{c}{2},$$

we have

$$\begin{aligned} u'(t) &\geq u'(t_4) + \tfrac{c}{2}(t - t_4) \geq -\sqrt{2E(t_4)} + \tfrac{c}{2}(t - t_4) \\ &\geq -\sqrt{2}c^{\gamma} + \tfrac{c}{2}(t - t_4). \end{aligned} \tag{2.52}$$

Therefore, there exists $t_5 \in [t_4, t_4 + 2\sqrt{2}c^{\gamma-1}]$ with $u'(t_5) = 0$ and $u'(s) \leq 0$ for all $s \in [t_4, t_5]$. Thus

$$\psi(t_5) = 2(k+1)\pi, \tag{2.53}$$

and $\psi(s) \in [2k\pi + \frac{3}{2}\pi, 2(k+1)\pi]$, for all $s \in [t_4, t_5]$. Furthermore,

$$\begin{aligned} u(t_5) = u(t_4) + \int_{t_4}^{t_5} u'(s)\,ds &\geq -\alpha c^{2\gamma-1} - \sqrt{2}c^{\gamma}2\sqrt{2}c^{\gamma-1} \\ &\geq -(\alpha+4)c^{2\gamma-1}. \end{aligned} \tag{2.54}$$

By (1.3) there exist constants $M_1 > 0$, and M_2 such that $G(u) \leq M_1 u^2 + M_2$ for $u \leq 0$. Let C_5 be such that for $c > C_5$ we have

$$G(-(\alpha+4)c^{2\gamma-1}) \leq M_1(\alpha+4)^2 c^{4\gamma-2} + M_2 \leq \frac{c^{2\gamma}}{2}. \tag{2.55}$$

Thus

$$G(u(t_4)) \leq G(u(t_5)) \leq M_1(u(t_5))^2 + M_2 \leq \frac{c^{2\gamma}}{2}. \tag{2.56}$$

Since $E(t_4) = c^{2\gamma}$, from (2.56) we have

$$u'(t_4) \leq -c^{\gamma}, \tag{2.57}$$

for c sufficiently large. By the continuity of u' we see that there exists $\tilde{t} \in (t_4, t_5)$ such that

$$u'(\tilde{t}) = -\frac{c^{\gamma}}{2}. \tag{2.58}$$

Since $|u'(t_4)| \leq \sqrt{2}c^{\gamma}$, and $|g(u)| \leq (M+1)|u|$ for $u \leq 0$ sufficiently large, then for $t \in [t_4, \tilde{t}]$ we have

$$u''(t) \leq \frac{8(N-1)\sqrt{Q}}{c^{\gamma-1}N}\sqrt{2}c^{\gamma} + 3c + (M+1)(\alpha+4)c^{2\gamma-1} \leq (16\sqrt{Q}+4)c, \tag{2.59}$$

for c sufficiently large. Thus by integrating (2.59) on $[t_4, \tilde{t}]$, and using (2.57) and (2.58) we have that $\tilde{t} - t_4 \geq c^{\gamma-1}/2(16\sqrt{Q}+4)$. Hence

$$\begin{aligned} u(t_5) \leq u(\tilde{t}) &= u(t_4) + \int_{t_4}^{\tilde{t}} u'(s)\,ds \\ &\leq 0 - \frac{c^{\gamma}}{2}(\tilde{t} - t_4) \leq -\frac{c^{2\gamma-1}}{64\sqrt{Q}+16}. \end{aligned} \tag{2.60}$$

From (2.54) and (2.60) we see that if $c > C_5$, where C_5 is in addition assumed to be such that (2.59) holds, then the lemma is proven.

The next lemma summarizes the results of this section.

Lemma 2.7. *Given any positive integer* j *there exists a real number* m_j *such that if* $c > m_j$, $E(t,d,c) > 0$ *for all* $t \in [0,T]$, $E(t_1,d,c) = c^{2\gamma}$, *for some* $t_1 \in [Nc^{\gamma-1}/4, T]$, $\gamma \in [(5\gamma_1 + 3\gamma_2)/8, (3\gamma_1 + 5\gamma_2)/8]$, *and either*

(A) $\psi(t_1,d,c) \in [2k\pi + \frac{\pi}{2}, 2k\pi + \frac{3}{2}\pi]$, *or*

(B) $\psi(t_1,d,c) \in [2k\pi, 2k\pi + \frac{\pi}{2}) \cup (2k\pi + \frac{3}{2}\pi, 2(k+1)\pi]$, *and* $u(t_1,d,c) \geq -c^{2\gamma-1}$,

then $\psi(T,d,c) > j\pi + \pi/2$.

Proof. We split the proof into four cases depending on the quadrant where $\psi(t_1,d,c)$ lies.

Case I. Suppose that $\psi(t_1,d,c) \in [2k\pi, 2k\pi + \pi/2]$. By (2.4) there exists C_5 such that for $c > C_5$ we have that $(2\sqrt{2} + 2\alpha\sqrt{N} + 2)c^{\gamma_2 - 1} < c^{-\varepsilon/2}$. Thus by applying consecutively Lemmas 2.2 (or 2.3), 2.4, 2.5, 2.6, we see that if $c > C := \max\{C_1, \dots, C_5\}$ then there exist $t_2 < t_3 < t_4 < t_5$ such that

$$t_5 \in [t_1, t_1 + k^* c^{-(\varepsilon/2)}] \tag{2.61}$$

and

$$\psi(t_5,d,c) = 2(k+1)\pi. \tag{2.62}$$

Moreover, from Lemma 2.2. we have

$$E(t_2,d,c) = c^{2\gamma'}, \tag{2.63}$$

with

$$-\frac{\ln 128}{2\ln c} + \gamma \leq \gamma' \leq \gamma + \frac{\ln Q}{2\ln c}. \tag{2.64}$$

Also, by combining (2.64) with the results from Lemmas 2.4 and 2.5 we see that $E(t_4,d,c) = c^{2\gamma''}$, with

$$\gamma - \frac{2\ln(3/4) + \ln 128}{2\ln c} \leq \gamma'' \leq \gamma + \frac{\ln 2 + \ln Q}{2\ln c}. \tag{2.65}$$

Now by Lemma 2.6 (since in this case $\alpha = 0$) we have that $u(t_5) = -c^{2\bar{\gamma}-1}$ with

$$\gamma - \frac{\ln(72(64\sqrt{Q} + 16))}{2\ln c} \leq \bar{\gamma} \leq \gamma + \frac{\ln 8Q}{2\ln c}, \tag{2.66}$$

or

$$\begin{aligned} -k'c^{2\gamma-1} &:= -8Qc^{2\gamma-1} \leq u(t_5) \leq -[72(64\sqrt{Q} + 16)]c^{2\gamma-1} \\ &:= -K'c^{2\gamma-1}. \end{aligned} \tag{2.67}$$

Let now

$$m_j := \max\left\{\exp\left(\frac{4j|\ln k'|}{3(\gamma_2 - \gamma_1)}\right), \exp\left(\frac{4j|\ln K'|}{3(\gamma_2 - \gamma_1)}\right), (j/T)^{2/\varepsilon}, C\right\}. \tag{2.68}$$

From (2.67) and (2.68) it follows that

$$(2.69)\qquad \frac{5\gamma_1+3\gamma_2}{8}-\frac{3\gamma_2-3\gamma_1}{8j}\le\bar\gamma\le\frac{3\gamma_1+5\gamma_2}{8}+\frac{3\gamma_2-3\gamma_1}{8j}.$$

Also, from (2.68) we have

$$(2.70)\qquad t_5-t_1\le T/j.$$

Now, we observe that $t_5=\tau$, and $\bar\gamma=\gamma$ satisfy the hypothesis of Lemma 2.3. Thus, iterating the above argument j times we see that there exist $t_5<t_6<\cdots<t_{j+3}<T$, with $\psi(t_{j+3},d,c)=2(k+j)\pi$. Hence, by Remark 2.1. we have that

$$(2.71)\qquad \psi(T,d,c)>2(k+j)\pi-\tfrac{\pi}{2}\ge 2j\pi-\tfrac{\pi}{2}\ge j\pi+\tfrac{\pi}{2},$$

and that proves Case I.

Case II. If $\psi(t_1,d,c)\in[2k\pi+\pi/2,(2k+1)\pi]$, then by applying Lemmas 2.4, 2.5, 2.6. we see that there exists t_5 that satisfies (2.61), (2.62), (2.67), (2.69), and thus, we are in Case I.

Case III. If $\psi(t_1,d,c)\in[(2k+1)\pi,(2k+1)\pi+\pi/2]$, then we can apply Lemmas 2.5 and 2.6, and Case III reduces to Case I.

Case IV. If $\psi(t_1,d,c)\in[(2k+1)\pi+\pi/2,2(k+1)\pi]$, then by applying Lemma 2.6 we are in Case I, and that concludes the proof of Lemma 2.7.

3. Energy analysis

Throughout this section we use K, k', K', $\bar k$, k_1, k_2 to denote various constants independent of (c,γ).

Let now $u(t):=u(t,d,c)$ be a solution to (1.8). Suppose that there exists $\hat t\in[Nc^{\gamma-1}/4,T]$, with $\gamma\in[\gamma_1,\gamma_2]$, such that

$$(3.1)\qquad E(\hat t,d,c)=c^{2\gamma},\quad E(t)>0\quad\text{for all } t\in[0,\hat t] \text{ and } u(\hat t)\ge -c^{2\gamma-1}.$$

Arguing as in the proof of Lemma 2.7 (using Lemmas 2.2, 2.4 and 2.5 if $\psi(\hat t,d,c)\in[2k\pi,2k\pi+\pi/2]$, Lemmas 2.4 and 2.5 if $\psi(\hat t,d,c)\in[2k\pi+\pi/2,(2k+1)\pi]$, Lemma 2.5 if $\psi(\hat t,d,c)\in[(2k+1)\pi,(2k+1)\pi+\pi/2]$, and Lemmas 2.6, 2.3, 2.4 and 2.5 if $\psi(\hat t,d,c)\in[(2k+1)\pi+\pi/2,2(k+1)\pi]$), we see that there exists $\tau_1\in[\hat t,\hat t+Kc^{-(\varepsilon/2)}]$ such that $u(\tau_1)=0$, $u'(\tau_1)<0$, and

$$(3.2)\qquad k'c^{2\gamma}\le E(\tau_1,d,c)\le K'c^{2\gamma}.$$

Now we prove by induction

Lemma 3.1. *Let $\hat t$ be as in (3.1) and let $\hat t\le\tau_1<\hat\tau_1<\tau_2<\cdots<\tau_i<\hat\tau_i<\tau_{i+1}<\cdots\le t$ denote the zeroes of $u(t)$ for $t>\tau_1$ such that $E(s,d,c)>0$ for $s\in[\tau_1,t)$. If for some $\kappa\in(0,1)$ $L(\kappa)=\infty$, then there exists c^* such that if $c>c^*$ then*

$$(3.3)\qquad \begin{aligned} k'c^{2\gamma+(\gamma-1)N} &\le \tau_i^N E(\tau_i,d,c)\\ &\le \tau_i^N[k''c^{2\gamma}+(i-1)c^{3\gamma-1}]\le K'c^{2\gamma}. \end{aligned}$$

In particular $E(t,d,c)>0$ for all $t\in[0,T]$.

Proof. Since $\tau_1 \geq \hat{t} \geq c^{\gamma-1}$ from (3.2) we have (3.3) for $i = 1$. Also, from (3.2) we see that

$$k' c^{2\gamma+(\gamma-1)N} \leq E(\tau_i, d, c) \leq K' c^{2\gamma}. \tag{3.4}$$

Thus, as in (2.54) we obtain the existence of $a \in [\tau_i, \tau_i + Kc^{\gamma-1}]$ such that $u'(a) = 0$ and $u(a) > -(\alpha+4)c^{2\gamma-1}$. Furthermore, since $u''(t) \leq Kc^{1-\gamma}c^{\gamma} + 3c + (M+1)|u(a)| \leq Kc$ for $t \in [\tau_i, a]$ we infer $0 = u'(a) \leq -c^{\gamma+(\gamma-1)(N/2)} + Kc(a - \tau_i)$. Therefore $a - \tau_i \geq Kc^{\gamma-1+(\gamma-1)(N/2)}$.

Also, by the continuity of u' we see that there exists $\hat{a} \in [\tau_i, a]$ such that $u'(\hat{a}) = \frac{1}{2}u'(\tau_i)$. Since $u'(\hat{a}) = u'(\tau_i) + \int_{\tau_i}^{\hat{a}} u''(s)\,ds \leq u'(\tau_i) + Kc(\hat{a} - \tau_i)$ we have $\hat{a} - \tau_i \geq Kc^{\gamma-1+(\gamma-1)(N/2)}$. Thus

$$u(a) \leq u(\hat{a}) = u(\tau_i) + u'(\hat{a})(\hat{a} - \tau_i) \leq -Kc^{2\gamma-1+(\gamma-1)N}.$$

Arguing as in (2.6)–(2.7) we see that $\hat{\tau}_i < a + Kc^{\gamma-1}$. In particular there exists a unique b (u is convex on $(\tau_i, \hat{\tau}_i)$) such that $u(b) = -c^{1/(1+\rho)}$. Since

$$u(b) = u(a) + \int_a^b u'(s)\,ds \leq -Kc^{2\gamma-1+(\gamma-1)N} + (b-a)Kc^{\gamma},$$

we infer

$$b - a \geq Kc^{\gamma-1+(\gamma-1)N}, \tag{3.5}$$

where we have also used the fact that $2\gamma + (\gamma-1)N - 1 > 1/(1+\rho)$. Hence, as in (2.12) we obtain that $E(\hat{\tau}_i, d, c) \geq Kc^{2\gamma+(\gamma-1)N}$. Imitating the arguments used in Lemma 2.4 and Lemma 2.5 we show the existence of $\hat{\tau}_i < \tau_i' < \tau_i^* < \tau_i'' < \tau_{i+1}$ such that $u(\tau_i') = u(\tau_i'') = Kg^{-1}(4c)$, $u'(\tau_i') > 0$, $u'(\tau_i'') < 0$, and $u'(\tau_i^*) = 0$. In addition, we obtain that

$$\tau_i' - \tau_i \leq Kc^{[1/(1+\rho)]-\gamma-(\gamma-1)(N/2)}, \tag{3.6}$$

and $E(\tau_i^*, d, c) \geq Kc^{2\gamma+(\gamma-1)N}$. Therefore, as in Lemma 2.4 we infer

$$\tau_{i+1} - \tau_i'' \leq Kc^{[1/(1+\rho)]-\gamma-(\gamma-1)(N/2)}. \tag{3.7}$$

Since $[1/(1+\rho)] - \gamma - (\gamma-1)(N/2) < \gamma + (\gamma-1)N - 1$ (see (2.4a)) we have

$$\left|\int_a^b Kcr^{N-1}u(r)\,dr\right| > \left|\int_{[\tau_i, \tau_i'] \cup [\tau_i'', \tau_{i+1}]} Kcr^{N-1}u(r)\,dr\right|. \tag{3.8}$$

Multiplying (1.8) by $r^N u'$ and integrating over $[\tau_i, \tau_{i+1}]$ we obtain

$$\begin{aligned} &\tau_{i+1}^N E(\tau_{i+1}, d, c) - \tau_i^N E(\tau_i, d, c) \\ &\quad + \int_{\tau_i}^{\tau_{i+1}} \left[\frac{N-2}{2} r^{N-1}(u'(r))^2 - Nr^{N-1}G(u(r))\right] dr \\ &= \int_{\tau_i}^{\tau_{i+1}} [c\varphi(r) + p(r)] r^N u'(r)\,dr, \end{aligned} \tag{3.9}$$

where we have integrated by parts the term $\int_{\tau_i}^{\tau_{i+1}} r^N[u''(r)u'(r) + (G(u(r)))']\,dr$. Similarly, multiplying (1.8) by $r^{N-1}u$ and integrating over $[\tau_i, \tau_{i+1}]$ we infer

$$\int_{\tau_i}^{\tau_{i+1}} r^{N-1}(u'(r))^2\,dr = \int_{\tau_i}^{\tau_{i+1}} r^{N-1}\{g(u(r))u(r) - [c\varphi(r) + p(r)]u(r)\}\,dr. \tag{3.10}$$

By replacing (3.10) in (3.9) we obtain

$$\begin{aligned} \tau_{i+1}^N E(\tau_{i+1}, d, c) - \tau_i^N E(\tau_i, d, c) &= \int_{\tau_i}^{\tau_{i+1}} r^{N-1}\left[NG(u(r)) - \frac{N-2}{2}g(u(r))u(r))\right] dr \\ &+ \int_{\tau_i}^{\tau_{i+1}} r^{N-1}[c\varphi(r) + p(r)]\left[ru'(r) + \frac{N-2}{2}u(r)\right] dr. \end{aligned} \tag{3.11}$$

Integrating by parts the last term we get

(3.12)

$$\begin{aligned} &\tau_{i+1}^N E(\tau_{i+1}, d, c) - \tau_i^N E(\tau_i, d, c) \\ &\geq \int_{\tau_i}^{\tau_{i+1}} r^{N-1}\left[NG(u(r)) - \frac{N-2}{2}g(u(r))u(r)\right] dr \\ &\quad - \frac{c}{2}\int_{\tau_i}^{\tau_{i+1}} \left(\frac{N}{2} + 1\right) r^{N-1}u(r)\,dr \\ &\geq \int_{[\tau_i, \tau_i'] \cup [\tau_i'', \tau_{i+1}]} r^{N-1}\left[NG(u(r)) - \frac{N-2}{2}g(u(r))u(r) - \bar{k}cu(r)\right] dr \\ &\quad + \int_{\tau_i'}^{\tau_i''} r^{N-1}\left[NG(u(r)) - \frac{N-2}{2}g(u(r))u(r) - \bar{k}cu(r)\right] dr \\ &\quad - \bar{k}c\int_a^b r^{N-1}u(r)\,dr. \end{aligned}$$

Since g is an increasing function for $u > 0$ we have

$$G(u) = \int_0^{\kappa u} g(s)\,ds + \int_{\kappa u}^{u} g(s)\,ds \geq G(\kappa u) + (1-\kappa)ug(\kappa u).$$

Thus, from the assumption that $L(\kappa) = \infty$, there exists $k_1 \geq 0$ such that for $u \geq k_1$ (k_1 is chosen so that $NG(\kappa u) - ((N-2)/2)g(u)u \geq 0$ for $\kappa u \geq k_1$)

$$\begin{aligned} NG(u) - \frac{N-2}{2}g(u)u &\geq NG(\kappa u) + N\left(\frac{1-\kappa}{\kappa}\right)\kappa ug(\kappa u) - \frac{N-2}{2}g(u)u \\ &\geq N\left(\frac{1-\kappa}{\kappa}\right)\kappa ug(\kappa u) \geq N\left(\frac{1-\kappa}{\kappa}\right)G(\kappa u) \\ &\geq \left(\frac{1-\kappa}{\kappa}\right)\frac{N-2}{2}g(u)u = k_2 g(u)u. \end{aligned} \tag{3.13}$$

Thus if c is large enough and τ_i', τ_i'' are chosen so that $g(u(\tau_i')) = g(u(\tau_i'')) = \bar{k}c/k_2$ then

$$\int_{\tau_i'}^{\tau_i''} r^{N-1}\left[NG(u(r)) - \frac{N-2}{2}g(u(r))u(r) - c\bar{k}u(r)\right]dr \geq 0. \tag{3.14}$$

From (1.3) it follows that $L(1, u)$ is bounded below. Thus, (3.8), (3.14) and (3.12) yield

$$\tau_{i+1}^N E(\tau_{i+1}, d, c) \geq \tau_i^N E(\tau_i, d, c) \geq Kc^{2\gamma+(\gamma-1)N}. \tag{3.15}$$

On the other hand, since φ is differentiable we have

$$\begin{aligned} E(\tau_{i+1}) &\leq E(\tau_i) + \int_{\tau_i}^{\tau_{i+1}} (c\varphi(s) + p(s))u'(s)\,ds \\ &\leq E(\tau_i) - c\int_{\tau_i}^{\tau_{i+1}} \varphi'(s)u(s)\,ds \leq E(\tau_i) + Kc\int_{\tau_i}^{\hat{\tau}_i} |u(s)|\,ds \\ &\leq E(\tau_i) + Kcc^{2\gamma-1}c^{\gamma-1}. \end{aligned}$$

Hence inductively we obtain

$$E(\tau_{i+1}) \leq E(\tau_1) + (i-1)Kc^{3\gamma-1}.$$

Since $\tau_{i+1} - \tau_i \geq Kc^{\gamma-1}$, we see that $i \leq (T/(Kc^{\gamma-1}))$. Therefore

$$E(\tau_{i+1}) \leq E(\tau_1) + Kc^{2\gamma} \leq K'c^{2\gamma}, \tag{3.16}$$

which together with (3.15) proves (3.3).

Let $\gamma_0 := (\gamma_1 + \gamma_2)/2$, where γ_1 and γ_2 are as in (2.3).

Lemma 3.2. *Let $\tau \in [0, T]$ be such that $u'(\tau) \geq 0$ and $u(\tau) \leq -c^{2\gamma_0 - 1}$. There exist C such that if $c > C$ then for some*

$$\tau_1 \in \left(\sqrt{2N}\left(\frac{|u(\tau)|}{3c + (M+1)|u(\tau)|}\right)^{(1/2)}, 2\sqrt{N}\left(\frac{|u(\tau)|}{c}\right)^{(1/2)}\right)$$

we have $u(\tau_1 + \tau) = 0$, $u(t) < 0$ for all $t \in [\tau, \tau + \tau_1]$, and $E(\tau_1, d, c) \geq c^{2\gamma_0}/64N$.

Proof. Since g is an increasing function, for $t > \tau$ we have

$$u'(t) \geq t^{-N+1}\frac{c}{2}\int_\tau^t s^{N-1}\,ds \geq \frac{c}{2N}(t - \tau).$$

Hence $u(t) \geq u(\tau) + (c/4N)(t-\tau)^2$. Therefore, $\tau_1 \leq 2\sqrt{N}(|u(\tau)|/c)^{(1/2)}$ exists such that $u(\tau_1 + \tau) = 0$. Similarly using that for $u < 0$ sufficiently large $|g(u)| \leq (M+1)|u|$ we have $u'(t) < t(3c + (M+1)|u(\tau)|)/N$. Thus, $u(t) \leq u(\tau) + t^2(3c + (M+1)|u(\tau)|)/(2N)$. This proves that

$$\tau_1 \geq \sqrt{2N}\left(\frac{|u(\tau)|}{3c + (M+1)|u(\tau)|}\right)^{(1/2)} \geq \frac{\sqrt{2N}}{4}c^{\gamma_0 - 1}.$$

Hence,

$$u'(\tau_1) \geq \frac{c - \|p\|_\infty}{4N} \sqrt{2N} c^{\gamma_0 - 1} \geq \frac{1}{4\sqrt{2N}} c^{\gamma_0}.$$

Therefore, $E(\tau_1, d, c) \geq c^{2\gamma_0}/64N$, and that concludes the proof of the lemma.

Lemma 3.3. *Let C be as in Lemma 3.2, and $c > C$. If $\bar{t} \in [0, T]$ is the largest number such that $E(\bar{t}, d, c) = c^{2\gamma_0}/64N$, and $E(t, d, c) > 0$ for all $t \in [0, \bar{t}]$ then $u(\bar{t}) \geq -c^{2\gamma_0 - 1}$.*

Proof. Suppose that $u(\bar{t}) < -c^{2\gamma_0 - 1}$. From Lemma 3.2 for all $t > \bar{t}$ with $u(t) < 0$ we have

$$u''(t) \geq -\frac{4(N-1)}{\sqrt{2N}} c^{1-\gamma_0} \frac{1}{4\sqrt{2N}} c^{\gamma_0} + c\varphi(t) + p(t) - g(u(t)) \geq \frac{c}{2}\left(1 - \frac{N-1}{2N}\right).$$

If $u'(\bar{t}) < 0$ we let t_0 denote the smallest number greater than $\bar{t}$ such that $u'(t_0) = 0$, and if $u'(\bar{t}) > 0$ we let $t_0 = \bar{t}$. Of course, $u(t_0) \leq u(\bar{t}) < -c^{2\gamma_0 - 1}$. Since for all $s > t_0$ with $u < 0$ on (t_0, s) we have that $|u'(s)| \leq c^{\gamma_0}/4\sqrt{2N}$, hence $u(t) < 0$ for $t \in (t_0, t_0 + 4\sqrt{2N} c^{\gamma_0 - 1})$. Thus, for $t = t_0 + 4\sqrt{2N} c^{\gamma_0 - 1}$ we have

$$\begin{aligned} u'(t_0 + 4\sqrt{2N} c^{\gamma_0 - 1}) &= u'(t_0) + \int_{t_0}^{t_0 + 4\sqrt{2N} c^{\gamma_0 - 1}} u''(s)\, ds \\ &\geq \frac{c}{2}\left(1 - \frac{N-1}{2N}\right) 4\sqrt{2N} c^{\gamma_0 - 1} > \frac{1}{4\sqrt{2N}} c^{\gamma_0}, \end{aligned}$$

which contradicts the assumption that $\bar{t}$ is the largest number such that $E(\bar{t}, d, c) = c^{2\gamma_0}/64N$ and $E(t, d, c) > 0$ for all $t \in [0, \bar{t}]$. Hence the lemma is proven.

Now we summarize the above results in the following lemma.

Lemma 3.4. *If $L(\kappa) = \infty$ for some $\kappa \in (0, 1)$, then there exists C^* such that if $c > C^*$, and $d \leq -c^{2\gamma_0 - 1}$ then $E(t, d, c) > 0$ for all $t \in [0, T]$.*

Proof. From Lemma 3.2 (taking $\tau = 0$) we see that either $E(t, d, c) \geq c^{2\gamma_0}/64N$ for all $t \in [0, T]$, or there exists $\bar{t} \geq \tau_1 > \frac{1}{4} c^{2\gamma_0 - 1}$ such that $E(\bar{t}, d, c) = c^{2\gamma_0}/64N$ and $E(t, d, c) > 0$ for all $t \in [0, \bar{t}]$. Also by Lemma 3.3 we have $u(\bar{t}) \geq -c^{2\gamma_0 - 1}$. Hence, taking $\gamma' = \gamma_0 - (\ln 64N)/(2 \ln c)$ we have $E(\bar{t}, d, c) = c^{2\gamma'}$, and

$$u(\bar{t}) \geq -c^{2\gamma_0 - 1} = -c^{2(\gamma_0 - \gamma')} c^{2\gamma' - 1} = -64N c^{2\gamma' - 1} \equiv -\alpha c^{2\gamma' - 1}.$$

Let now $\bar{c}$ be such that if $c > \bar{c}$ then $(\ln 64N)/(2\ln c) < \frac{1}{4}(\gamma_2 - \gamma_1)$. By applying consecutively the lemmas from section 2, depending on the quadrant where $u(\bar{t})$ lies we obtain the existence of $\tau_1 \in [\bar{t}, \bar{t} + Kc^{-(\varepsilon/2)}]$ such that $u(\tau_1) = 0$, $u'(\tau_1) < 0$ and (3.2) holds. Now, directly from Lemma 3.1 by taking $C^* = \max\{c^*, C, \bar{c}\}$ we obtain the proof of the lemma.

4. Separation of zeroes

The following lemma is a version of the Sturm comparison theorem for singular linear differential equations, and it proves our main estimate on the separation of zeroes.

Lemma 4.1. *There exists* $H := H(c) > 0$ *such that if* u *satisfies* (1.8), $u(t^0) < -H$ *and* $u'(t^0) = 0$, *for some* $t^0 \in [0, T]$, *then* $u(t) < 0$ *for all* $t \in (t^0, t^0 + \frac{1}{2}(M+1)^{-1/2})$. *Moreover, if for some* $t^0 \in [0, T]$, $u'(t^0, d, c) = 0$ *and* $u(t^0, d, c) < -H$, *then*

$$\psi(T, d, c) \le \psi(t^0, d, c) + \pi/2 + 2J(t^0)\pi,$$

where $J(t^0)$ *is the greatest integer less then* $2(M+1)^{1/2}(T - t^0)$. *In particular taking* $t^0 = 0$ *we have*

$$\limsup_{-d\to\infty} \psi(T, d, c) \le \frac{\pi}{2} + 2J(0) := S.$$

Proof. From (1.3)–(1.4) we see that there exists a real number $M^* \le 0$ such that

$$g(s) - (M+1)s \ge M^* \quad \text{for all } s \in \mathbf{R}. \tag{4.1}$$

Thus, for $A(t) := p(t) + c\varphi(t) - g(u(t)) + (M+1)u(t)$, we have

$$A(t) \le \|p\|_\infty + c\|\varphi\|_\infty - M^* := H. \tag{4.2}$$

By rewriting (1.8) we obtain

$$\begin{gathered} u''(t) + \frac{N-1}{t}u'(t) + (M+1)u(t) = A(t), \\ u(t^0) = -h, \quad u'(t^0) = 0, \end{gathered} \tag{4.3}$$

where $h \in \mathbf{R}$ is such that $h > H$. Let now

$$F(t) = \tfrac{1}{2}[(u'(t))^2 + (M+1)u^2(t)]. \tag{4.4}$$

From (4.3) and (4.4) we see that

$$dF/dt \le A(t)u'(t) \le H(2F(t))^{1/2}. \tag{4.5}$$

Integrating (4.5) on $[t^0, t]$ we obtain

$$\begin{aligned} |u'(t)| &\le (2F(t))^{1/2} \le (2F(t^0))^{1/2} + H(t - t^0) \\ &\le (M+1)^{1/2}h + H(t - t^0). \end{aligned} \tag{4.6}$$

Let $\tau > t^0$ be such that $u(\tau) = 0$, and $u(t) < 0$ for all $t \in [t^0, \tau]$. Hence, from (4.6) we have

$$\begin{aligned} 0 = u(\tau) &= u(t^0) + \int_{t^0}^{t} u'(s)\,ds \\ &\le u(t^0) + (M+1)^{1/2}h(t - t^0) + \tfrac{1}{2}H(t - t^0)^2 \\ &\le h[-1 + (M+1)^{1/2}(t - t^0) + \tfrac{1}{2}(t - t^0)^2]. \end{aligned} \tag{4.7}$$

Therefore

$$t - t^0 \le \tfrac{1}{2}(M+1)^{-1/2} := \Lambda . \tag{4.8}$$

Hence, $u(t) < 0$ for all $t \in [t^0, t^0 + \Lambda]$, and thus in particular

$$\psi(t^0 + \Lambda, d, c) \le \psi(t^0, d, c) + \pi/2 . \tag{4.9}$$

Iterating this argument we see that

$$\psi(t^0 + 2\Lambda, d, c) \le \psi(t^0, d, c) + \pi/2 + 2\pi , \tag{4.10}$$

and

$$\psi(t^0 + i\Lambda, d, c) \le \psi(t^0, d, c) + \pi/2 + (i-1)2\pi , \tag{4.11}$$

with $i \in \{0, 1, \dots, J(t^0)\}$. In particular

$$\begin{aligned} \psi(T, d, c) &\le \psi(t^0 + J(t^0)\Lambda, d, c) + 2\pi \\ &\le \psi(t^0, d, c) + \pi/2 + (J(t^0) - 1)2\pi + 2\pi \\ &= \psi(t^0, d, c) + \pi/2 + J(t^0)2\pi , \end{aligned} \tag{4.12}$$

and that proves the lemma.

5. Proof of Theorem A

By Lemma 3.4 we see that there exists C^* such that if $c > C^*$ and $d \le -c^{2\gamma_0 - 1}$ then $E(t, d, c) > 0$ for all $t \in [0, T]$. Therefore, if $c > m_j$ then by Lemma 2.7 we have that

$$\psi(T, -c^{2\gamma_0 - 1}, c) \ge j\pi + \pi/2 . \tag{5.1}$$

Let $J := [(S - \pi/2)/\pi] + 1$, where $[x]$ denotes the largest integer less than or equal to x. If $j \ge J$ then $j\pi + \pi/2 > S$. Hence by Lemma 4.1, (5.1), and the intermediate value theorem if $c > C_j = \max\{m_j, C^*\}$ then there exist numbers $d_J < d_{J+1} < \cdots < d_j$, such that

$$\psi(T, d_i, c) = i\pi + \pi/2, \qquad i \in \{J, J+1, \dots, j\} . \tag{5.2}$$

Hence, $u_i(x) = u(\|x\|, d_i, c)$ is a solution to (1.1) with exactly i interior nodal surfaces. Thus, Theorem A is proven.

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